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An analytical estimate of the period for the delayed logistic application and the Lotka-Volterra system

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Abstract. We first introduce a simple and new method for the quantitative analysis of some nonlinear oscillating systems. It is shown that if the dynamics of the system reduces to piecewise exponential growth and exponential damping phases, then the amplitude and period of the motion can be computed with accuracy in the nonlinear regime without invoking linear stability arguments or perturbative expansions. This method is then successfully applied to the delayed logistic application and to the Lotka-Volterra prey-predator model. For both of these systems, we provide an accurate analytical expression for the period of the oscillations in the nonlinear regime.

PACS. 47.20.Ky Nonlinearity (including bifurcation theory)

1 Introduction

This paper is concerned with the quantitative estimation of the temporal features of nonlinear oscillating systems. Among these, we particularly focus our attention on two paradigms for oscillations controlled by nonlinear effects, namely the delayed logistic application and the Lotka-Volterra prey-predator model.

Considering the evolution of a single population able to reproduce itself, Verhulst [16] was the first to realize that the growth of a population in a bounded domain could not continue indefinitely at the same rate and that some nonlinear damping should occur; he formalized this idea in the since then celebrated logistic equation which incorporates the first order nonlinear correction to the pure exponential growth of the population size $x(t)$:

$$\frac{dx}{dt} = x[1-x]. \qquad (1)$$

This model has its equivalent in the context of hydrodynamic instabilities for the square of the amplitude disturbances in a flow [7].

It was further realized that the nonlinear damping in (1) could for some reasons incorporate historical terms and that the growth of the population size $x(t)$ at time t could be a function of the population size at previous instants of time $x(t-t')$. The relative influence of the different time lags t' is weighted by a memory function $f(t')$ such that

$$\frac{dx}{dt} = x(t)\left[1-\int_0^t x(t-t')f(t')\,dt'\right]. \qquad (2)$$

In practice, the function $f(t')$ is peaked around a certain time lag τ, and in the limit where $f(t')$ is a pure delay (*i.e.* $f(t')=\delta(t'-\tau)$ where $\delta(t)$ is the Dirac delta function), equation (2) reduces to the delayed logistic equation

$$\frac{dx}{dt} = x[1-x(t-\tau)], \qquad (3)$$

which displays nonlinear oscillations for $\tau > \pi/2$. Equation (3) is sometimes referred to as the Hutchinson equation, Cherwell-Wright equation, or NLDS (Non Linear Delayed Saturation) model, depending on the community. This formulation is particularly widespread in the population dynamics community because the origin of the time delay τ receives there a clear physiological interpretation. It is usually attributed to an incubation period [15] when diseases leading to death affect the population growth, or may also represent a maturity time of the individuals in the population (see [13] and references therein). More generally, time lags come into play in various evolution equations (where they are usually included on a heuristic basis) when the dynamics of the system is ruled by two very different timescales: a short timescale characteristic of the growth rate and a long time characteristic of the feedback. Several examples can be found in different area including physiology [2,11], economics [5],

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astrophysics [20], combustion [6,12], optics [8], geophysics [1] or hydrodynamics [17,18]; all of them display nonlinear oscillations on which more or less chaotic fluctuations are superimposed. The work of Joulin [6] is particularly relevant in the sense that equation (3) is derived by an asymptotic expansion from the local equations of hydrodynamics to describe the oscillations of the standoff distance of a premixed planar flame front held downstream of a flat porous burner. The delay is here shown to be due to the slow, diffusive transport of heat from the flame front to the burner.

The effect of the interaction between different species has also been addressed. The simplest prey-predator model was first proposed by Lotka [9] in the context of competing chemical reactions with coupled second order kinetics, and then systematically investigated by Volterra [19], invoking the situation of two species in a closed space, one of them serving of food for the other ("*Deux espèces dont l'une dévore l'autre*" following the title of one of the chapters of his 1931 book). If $x(t)$ and $y(t)$ represent the populations of two species coexisting in a closed area, one of them (x) living on an infinite substrate and being the food of the other (y), this model writes, with appropriate amplitude and time rescalings as:

$$\frac{dx}{dt} = \delta\, x[1-y] \quad \text{and} \quad \frac{dy}{dt} = -\frac{1}{\delta}\, y[1-x] \tag{4}$$

where δ is a constant. This system displays nonlinear oscillations (see for example [13]), the amplitude and phase shift between $x(t)$ and $y(t)$ being given by the initial conditions $x(0)$, $y(0)$ and δ.

Although systems (3, 4) are widely commented on and referenced in the literature, including review monographs directly devoted to the subject [10,13,14], attempts to estimate the period of oscillation far beyond the vicinity of the onset of the oscillations have been made very recently only.

Taking advantage of the "exponential oscillation" like behavior of (3) far from the oscillation threshold ($\tau > \pi/2$), Villermaux and Hopfinger [18] derived an accurate expression for the oscillation period for a variant of the logistic equation, namely the NLDS model, which reduces to the logistic equation when the fluctuating quantity is positive. This expression proved to be particularly useful for extrapolation purposes in situations where τ can be computed a-priori from the physics of the problem. The period was found to be always larger than the delay τ, with a correction involving a rapidly varying function of τ, explaining why perturbative methods around the oscillation threshold failed at giving a correct estimate (see [3,13]). The derivation has been extended by Villermaux [17] for two coupled delayed logistic equations allowing to make, again, a direct comparison with a physical experiment, and original predictions.

The purpose of this work is to improve the 'one parameter free' derivation of reference [18] with a simple and general method. This method is presented in Section 2 and then applied to the delayed logistic application (Sect. 3) and to the Lotka-Volterra system (Sect. 4).

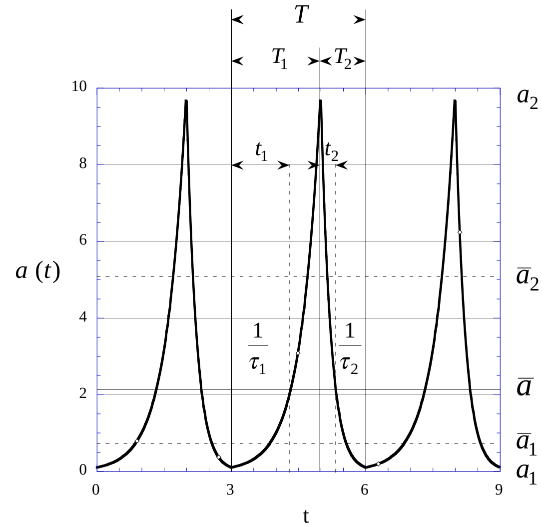

Fig. 1. Example of an exponential oscillating signal.

2 The relaxation oscillation method (ROM)

For the delayed logistic application and the Lotka-Volterra system, the ascending and descending part of the oscillations can be approximated by exponentials, in the nonlinear regime. The first step of the method is to study a pure exponential oscillating function $a(t)$:

$$a(t) = \begin{cases} a_1 e^{t/\tau_1} & \text{for} \quad 0 < t < T_1 \\ a_2 e^{-t/\tau_2} & \text{for} \quad T_1 < t < T_1 + T_2. \end{cases} \tag{5}$$

This function is presented in Figure 1 [1]. According to (5), $a(t)$ increases from the minimum a_1 to the maximum a_2 with a characteristic time τ_1, in the ascending phase. This growth is T_1 long and t_1 after having reached a_1, $a(t)$ reaches its mean value $\overline{a}$. In the descending part, $a(t)$ decreases from a_2 to a_1 with a characteristic time τ_2. This decrease is T_2 long and t_2 after the amplitude $a(t)$ had reached a_2, it reaches $\overline{a}$. The quantities of interest for the nonlinear oscillations are the maxima a_1, a_2 and the period $T = T_1 + T_2$. Using the exponential character of the oscillations equations (6–8) express these quantities as a function of the mean values $\overline{a}$, $\overline{a_1}$ and $\overline{a_2}$ and as a function of the characteristic times τ_1 and τ_2. Equation (6) relates the extrema to the mean value:

$$\frac{a_2 - a_1}{\ln(a_2/a_1)} = \overline{a}. \tag{6}$$

Considering the portion of $a(t)$ above and below $\overline{a}$, we define an upper and lower mean, referred to as $\overline{a_2}$, and $\overline{a_1}$ respectively. These mean values are related to the extrema

[1] Even if the function $a(t)$ is continue, its first derivative shows discontinuities at the maxima (a_1 and a_2). It follows that $a(t)$ cannot be the exact solution of an oscillating system described by a set of differential equations. This does not prevent it to be, in some cases, a good approximation of such a system.

by the same type of equation as (6):

$$\frac{\overline{a}-a_1}{\ln(\overline{a}/a_1)} = \overline{a_1} \quad \text{and} \quad \frac{a_2-\overline{a}}{\ln(a_2/\overline{a})} = \overline{a_2}. \tag{7}$$

Equations (6, 7) are used to compute the extrema and the sub periods T_1 and T_2:

$$\frac{T_1}{\tau_1} = \frac{T_2}{\tau_2} = \ln(a_2/a_1). \tag{8}$$

Finally, the period of the signal, T, is determined from the extrema and from the characteristic times:

$$T = (\tau_1+\tau_2)\ln(a_2/a_1). \tag{9}$$

In the following, we will also use the expressions of the intermediate times t_1 and t_2 which separate respectively a_1 and a_2 from $\overline{a}$.

$$\frac{t_1}{\tau_1} = \frac{T_2-t_2}{\tau_2} = \ln\left(\frac{\overline{a}}{a_1}\right)$$

$$\frac{t_2}{\tau_2} = \frac{T_1-t_1}{\tau_1} = \ln\left(\frac{a_2}{\overline{a}}\right)\cdot \tag{10}$$

3 The delayed logistic application

In a dimensionless form, the delayed logistic application (3) can be written:

$$\frac{d\ln(x)}{dt} = 1 - x(t-\tau). \tag{11}$$

The minimum set of parameters which controls the evolution of $x(t)$, includes τ and the knowledge of x during the period $t \in [0,\tau]$. Equation (11) can then be used to obtain the evolution of x at later times $t > \tau$. In the following, we use the condition $x(t) = \beta$ for $0 < t < \tau$ where the constant β is set to 0.99 and can be considered as a small perturbation of the steady state $x = 1$. The maximal amplitudes and the period at the limit cycle in the nonlinear regime are not sensitive to the choice of the initial conditions; they only depend on the delay τ. According to the linear stability analysis, the steady state $x(t) = 1$ is stable for $\tau < \pi/2$ and exhibit limit cycles above the bifurcation value $\tau_c = \pi/2$. Near the bifurcation, the period of the oscillations is 2π [13]. The distance from the linear regime is measured by $\Delta\tau = \tau - \tau_c$.

A typical evolution of $x(t)$ and $x(t-\tau)$ is shown on a semi-logarithmic scale in Figure 2 for $\Delta\tau = 1$ ($\tau \approx 2.57$). The oscillations are strongly nonlinear and exhibit a "quasi exponential" behavior in the ascending and descending phases, which legitimate the method proposed. According to (11), the maxima x_1 and x_2 are reached when $x(t-\tau) = 1$. The ascending phase of x , corresponds to the region where $x(t-\tau) < 1$, and the descending phase, corresponds to the domain where $x(t-\tau) > 1$. The threshold value $x = 1$ is the average of x over a period.

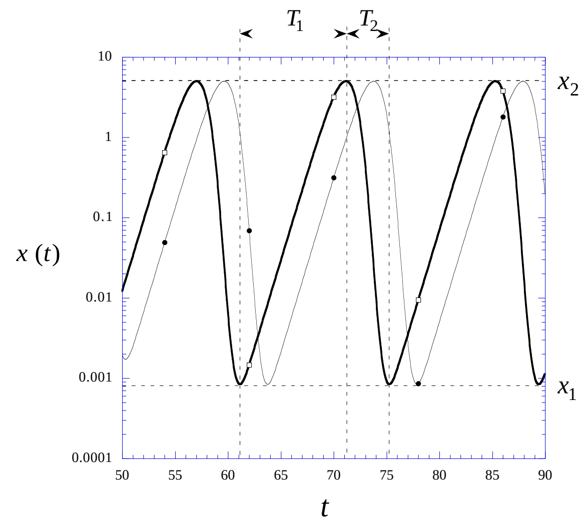

Fig. 2. Solution of the delayed logistic application obtained with $\Delta\tau = 1$; (□): $x(t)$; (●): $x(t-\tau)$.

Indeed, if $x(t)$ oscillates and remains strictly positive, $\ln(x(t))$ also oscillates with the same period. The average of the derivatives over a period is by definition null, $\overline{\frac{dx}{dt}} = \overline{\frac{d\ln(x)}{dt}} = 0$, which implies, using (11):

$$\overline{1-x(t-\tau)} = 0 \ \text{ implying } \ \overline{x(t-\tau)} = \overline{x} = 1. \tag{12}$$

The mean value of $x(t-\tau)$ during the ascending phase of x is thus $\overline{x_1}$ and its mean value during the descending phase of x is $\overline{x_2}$. In the nonlinear regime, the delayed logistic application can thus be approximated by:

$$x(t) = \begin{cases} \dfrac{dx}{dt} = +\dfrac{x}{\tau_1} & \text{with } \dfrac{1}{\tau_1} \equiv (1-\overline{x_1}) \quad \text{(GP)} \\ \dfrac{dx}{dt} = -\dfrac{x}{\tau_2} & \text{with } \dfrac{1}{\tau_2} \equiv (\overline{x_2}-1) \quad \text{(DP).} \end{cases} \tag{13}$$

Where (GP) and (DP) respectively stand for Growth Phase and Decaying Phase. From equation (6), we get the relation between the maxima:

$$x_1 - \ln(x_1) = x_2 - \ln(x_2) \ \text{ or } \ F(x_1) = F(x_2). \tag{14}$$

Where the function $F(x) = x - \ln(x)$ is a key function in this paper and is discussed in Appendix. Using (13, 14) with (8), the period T of the oscillations, can be expressed as a function of the maxima:

$$T = T_1 + T_2$$

with

$$T_1 = \frac{F(x_2)(x_2-x_1)}{F(x_2)-1}$$

$$T_2 = \frac{\ln(x_2)(x_2-x_1)}{F(x_2)-1}\cdot \tag{15}$$

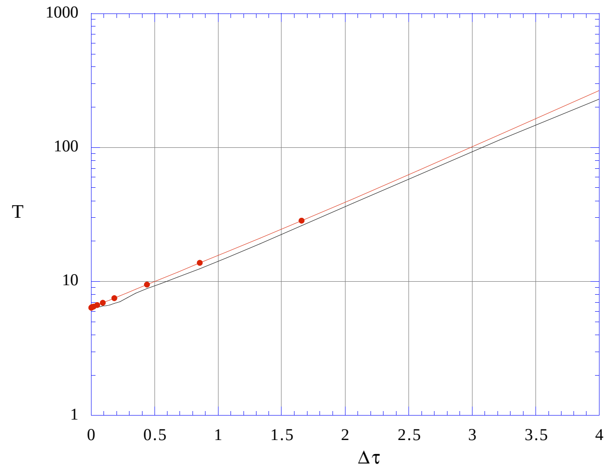

Fig. 3. Comparison of the period, T_{ni}, obtained by numerical integration (—) with the expression $e^{\tau} + \tau$ obtained with the ROM method (–●–).

According to equation (11), it takes a time $t = \tau$, to go from the mean value $x = 1$, to the maxima $x = x_1$, or $x = x_2$. With the notations of Section 2, $\tau = T_1 - t_1 = T_2 - t_2$. This equation, together with (7, 8, 13), leads to a relation between the maxima x_1, x_2 and the delay τ:

$$\frac{-\ln(x_2)\ln(x_1)}{F(x_1) - 1} = \tau. \qquad (16)$$

In the nonlinear regime, this expression can be simplified using the limits $x_1 \ll x_2$ and $x_2 \gg 1$. The first limit reduces equation (14) to $-\ln(x_1) \approx x_2 - \ln(x_2)$. The second leads to $F(x_2) \approx x_2$ (see Appendix). Equation (16) thus gives the evolution of the maximum as function of the delay: $x_2 \approx e^{\tau}$.

From the relations (15), one deduces:

$$T_1 \approx e^{\tau} \ , \ T_2 \approx \tau \quad \text{and} \quad T \approx e^{\tau} + \tau. \qquad (17)$$

Equation (17) is the central result of this section. It expresses the period of the delayed logistic application when the motion is strongly nonlinear. Figure 3 presents the variation with $\Delta\tau$ of the period, T_{ni}, obtained by numerical integration and compares it to the expression obtained in equation (17). In the range $10^{-4} < \Delta\tau < 4$, Table 1 also presents the comparison between the period obtained by a numerical integration (T_{ni}) of the system (11) and the estimation obtained with (17). The agreement between T_{ni} and the ROM estimation is within 10% for $0 < \Delta\tau < 3$.

4 The Lotka-Volterra system

In a dimensionless form, the Lotka-Volterra system can be written:

$$\begin{cases} \dfrac{d\ln(x)}{dt} = +\delta(1-y) \\[2ex] \dfrac{d\ln(y)}{dt} = -\dfrac{1}{\delta}(1-x). \end{cases} \qquad (18)$$

Table 1. Comparison between the period obtained with the ROM method ($e^{\tau} + \tau$) and the period obtained by numerical integration (T_{ni}).

$\Delta\tau$	τ	T_{ni}	$e^{\tau} + \tau$	Error (%)
1.00e-04	1.57	6.28	6.38	1.7
0.00129	1.57	6.28	6.39	1.8
0.0116	1.58	6.30	6.45	2.3
0.104	1.67	6.58	7.01	6.6
0.447	2.02	8.89	9.54	7.3
1.34	2.91	19.3	21.2	10.2
1.93	3.50	33.5	36.5	8.9
2.78	4.35	75.2	81.6	8.5
4.00	5.57	231	268	16.1

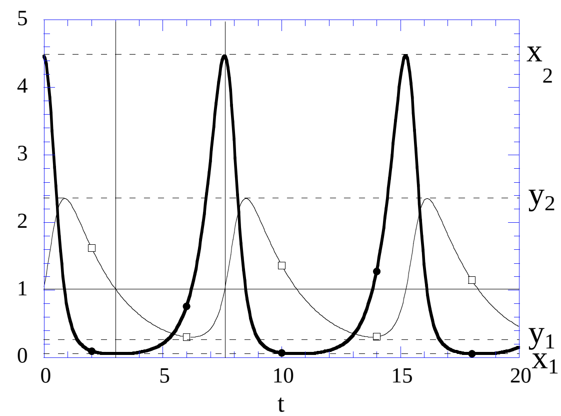

Fig. 4. Evolution of preys (x, –●–) and predators (y, –□–) for $\delta = 2$ and $H = 1.4H_{min}$.

This system is conservative in the sense that it admits a first integral:

$$H = F(x) + \delta^2 F(y) \quad \text{where}$$
$$F(\xi) = \xi - \ln(\xi) \quad \text{and} \quad H = const. \qquad (19)$$

The quantity H is thus conserved along the different trajectories in the (x, y) plane. The function F is always positive and admits a minimum $F(1) = 1$ (See Appendix). This implies that H is always larger than $H_{min} = 1 + \delta^2$. According to equation (19) any trajectory is defined by two parameters, δ and H. The first one expresses the asymmetry between the two populations and the second reflects the initial condition ($H = F(x_0) + \delta^2 F(y_0)$) giving an information on the strength of the nonlinearity (the larger $H - H_{min}$ the stronger the nonlinearity).

An example of solution is presented in Figure 4 with the parameters $\delta = 2$ and $H = 1.4\, H_{min} = 7$. Considering the evolution of preys x, we get from (18) that the maxima x_1 and x_2, are reached for $y = 1$ (in the same way, if $x = 1$ then $y = y_1$ or $y = y_2$). The decrease of x and its growth, correspond respectively to $y > 1$ and $y < 1$. Symmetrically, y grows for $x < 1$ and decays for $x > 1$.

The critical value, 1, is the average of the two populations over a cycle. This can be shown, using equations (18, 19) . The trajectories described by equations (19) are closed loops around the singularity $x = y = 1$ [19] and the populations x and y never relax to 0 (except for the trivial fixed point $\{x_0 = 0,\ y_0 = 0\}$). Then $\ln(x)$ and $\ln(y)$ oscillate with the same period as x and y and the average of their derivative over a period is 0. From equations (18) we have:

$$\left.\begin{array}{l}\overline{\delta(1-y)}=0\\[2ex] \overline{(x-1)/\delta}=0\end{array}\right\}\quad \text{or equivalently}\quad \overline{y}=\overline{x}=1. \qquad (20)$$

In other words, the mean value of y during the ascending phase of x is $\overline{y_1}$. From (18), we get that in the ascending phase, $\frac{dx}{dt} \approx \delta x(1-\overline{y_1})$, which describes an exponential. Using the same scheme for the descending phase of x and for the evolution of y, we get the approximate form of the Lotka-Volterra system:

$$x(t) = \begin{cases} \dfrac{dx}{dt} = +\dfrac{x}{\tau_{x1}} \quad \text{with} \quad \dfrac{1}{\tau_{x1}} \equiv \delta(1-\overline{y_1}) \quad \text{(GP)} \\[2ex] \dfrac{dx}{dt} = -\dfrac{x}{\tau_{x2}} \quad \text{with} \quad \dfrac{1}{\tau_{x2}} \equiv \delta(\overline{y_2}-1) \quad \text{(DP)} \end{cases} \qquad (21)$$

$$y(t) = \begin{cases} \dfrac{dy}{dt} = +\dfrac{y}{\tau_{y1}} \quad \text{with} \quad \dfrac{1}{\tau_{y1}} \equiv \dfrac{(\overline{x_2}-1)}{\delta} \quad \text{(GP)} \\[2ex] \dfrac{dy}{dt} = -\dfrac{y}{\tau_{y2}} \quad \text{with} \quad \dfrac{1}{\tau_{y2}} \equiv \dfrac{(1-\overline{x_1})}{\delta} \quad \text{(DP).} \end{cases} \qquad (22)$$

Considering first the population of preys, the maximal amplitude x_2 is related to the constant H *via* the first integral (19): $F(x_2) + \delta^2 = H$ which leads to the solution:

$$x_2 = F^{-1}_{>1}(1+\Delta H) \ \text{ with}$$
$$\Delta H = H - H_{min} = H - (1+\delta^2). \qquad (23)$$

The functions $F^{-1}_{>1}(\alpha)$ and $F^{-1}_{<1}(\alpha)$ stand respectively for the solution $x > 1$ and $x < 1$ of the equation $F(x) - \alpha = 0$ (see Appendix).

According to Section 2, the maxima x_1 and x_2 are related through equation (6) which can be written (using $\overline{x} = 1$):

$$F(x_1) = F(x_2). \qquad (24)$$

Obviously, the same equation applies for y:

$$F(y_1) = F(y_2). \qquad (25)$$

Using the notations of Section 2, in the ascending phase of y, it takes t_{2x} for $y(t)$ to go from $\overline{y} = 1$, to $y = y_2$. This can also be written $y_2 = 1 \times e^{t_{2x}(\overline{x_2}-1)/\delta}$. Using equation (7) for $\overline{x_2}$ and equation (10) for t_{2x} we get:

$$F(y_2) - 1 = \frac{1}{\delta^2}(F(x_2) - 1). \qquad (26)$$

It is worth noticing that equations (24–26) which have been obtained with the ROM method also constitute the exact solutions of the Lotka-Volterra system, according to the first integral (19). Since the maximal amplitudes can be obtained exactly from equation (19), the main interest of the ROM method for the Lotka-Volterra system is thus to provide a precise estimation of the period of the oscillations.

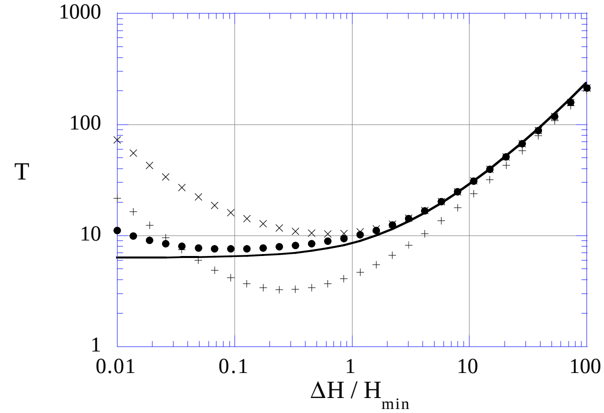

Fig. 5. Variation of the period of oscillation T as a function of $\Delta H/H_{min}$ for $\delta = 1$ (numerical integration — equation (28) - 0th order $+$, equation (28) - 1st order $\times$, equation (28) - 2nd order $\bullet$).

Using equation (25) in the expression of the period (9), we get $T = (\tau_{y1} + \tau_{y2})(y_2 - y_1)$. The characteristic times given by (22) reduce this expression to:

$$T = \delta \frac{(x_2 - x_1)(y_2 - y_1)}{F(x_2) - 1}. \qquad (27)$$

The expressions of the maxima (Eqs. (23–26)), reduce the expression of the period T, to a function of δ and H:

$$T = \frac{\delta}{\Delta H}\mathcal{F}(1+\Delta H)\mathcal{F}(1+\Delta H/\delta^2)$$
$$\text{where} \quad \mathcal{F}(x) = F^{-1}_{>1}(x) - F^{-1}_{<1}(x). \qquad (28)$$

This expression of the period is compared in Figures 5 and 6 to the value obtained by numerical integration of the system (4). Figure 5 presents the function $T(\Delta H/H_{min})$ for $\delta = 1$ and Figure 6 presents the function $T(\delta)$ for $\Delta H/H_{min} = 0.5$.

We first focus on Figure 5 and discuss the difference between the numerical integration and the value of T obtained with equation (28), using different methods to evaluate $F^{-1}_{>1}$ and $F^{-1}_{<1}$. The parameter $\Delta H/H_{min}$ used to present the evolution of the period, enables the separation between the linear and nonlinear domains: for $\Delta H/H_{min} < 1$, the period of the oscillations remains close to the linear limit 2π whereas its value increases strongly for $\Delta H/H_{min} > 1$.

The zero, first and second approximations presented in Figure 5 stand for the different order approximation of $F^{-1}_{>1}$ and $F^{-1}_{<1}$ presented in Appendix. For the whole range of $\Delta H/H_{min}$, the second order approximation remains close to the numerical solution. The maximum error is made in the linear domain (27%), where, however,

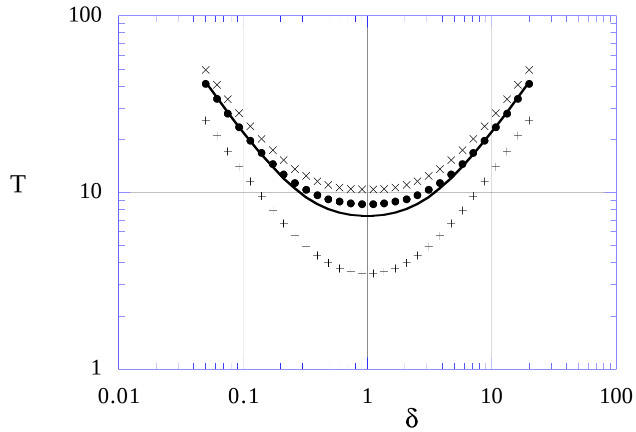

Fig. 6. Variation of the period of oscillation T as a function of $\delta = 1$ for $\Delta H/H_{min} = 0.5$ (numerical integration — equation (28) - 0th order +, equation (28) - 1st order ×, equation (28) - 2nd order •).

the ROM method is not supposed to apply [2]. The error is reduced to 20% as soon as $\Delta H/H_{min} > 0.5$, reaches 10% for $\Delta H/H_{min} > 1.7$ and is less than 1% for $\Delta H/H_{min} > 8$. Considering the first order approximation, the error is of the order of 20% for $\Delta H/H_{min} \approx 1$ and is reduced to 1% as soon as $\Delta H/H_{min} \approx 2$. The error made with the zero order approximation was always larger than 20% for the whole range of $\Delta H/H_{min}$. Using the first order approximation, the period can be written explicitly as function of δ and ΔH:

$$T_{a1} \approx \left(\frac{H}{\delta}+\frac{\delta}{\Delta H}\right)\left[1+\frac{\ln(1+\Delta H)}{\Delta H}+\frac{\ln(1+\Delta H/\delta^2)}{\Delta H/\delta^2}\right]. \tag{29}$$

A direct comparison of this formula with the calculated period is displayed in Table 2: all the errors reported, concern the special case $\delta = 1$. The study of the influence of δ for a fixed value of $\Delta H/H_{min}$ is presented in Figure 6. The period T reaches a minimum value for $\delta = 1$, where the error of the different approximations is maximum. This means that all the errors reported above for the case $\delta = 1$ will be reduced for any other value of δ. According to Figure 6, the maximum error of the second order approximation is 20% for $\delta = 1$ and is reduced to 10% for $\delta > 4$ or $\delta < 0.25$. The zero order approximation always underestimates the period whereas the first order always overestimates it. For the special case $\Delta H/H_{min} = 0.5$ the error made with the first order approximation goes down to 20% for $\delta > 10$ or $\delta < 0.1$. The estimation of the period with the second order approximation for any value of δ is better than 20% for $\Delta H/H_{min} > 0.5$ and better than 10% for $\Delta H/H_{min} > 1.7$. The accuracy of equation (29) is better than 20% for all δ, as soon as $\Delta H/H_{min} > 1$.

[2] In the linear regime, $x(t)$ and $y(t)$ are closer to a sinusoidal shape than to an exponential.

Table 2. Comparison between the period obtained with the ROM method and the period obtained by numerical integration (T_{ni}).

$\Delta H/H$	T_{ni}	T_{a1}	Error (%)
0.12689	6.5520	17.234	163.03
0.45203	7.2620	9.7233	33.893
0.62101	7.6440	9.3089	21.780
0.85317	8.1740	9.3013	13.792
1.1721	8.9180	9.6845	8.5955
1.6103	9.9520	10.477	5.2763
2.2122	11.380	11.734	3.1084
3.0392	13.328	13.551	1.6717
4.1753	15.966	16.076	0.68985
5.7361	19.520	19.525	0.024057

5 Conclusion

We have considered the oscillations of the delayed logistic application and the Lotka-Volterra system in the nonlinear regime. In that domain, we have shown that the approximation of the evolution by piecewise exponential growth and exponential damping phases provides an analytical expression for the period of the oscillations which remains accurate even far in the nonlinear domain (ROM method).

It is to be emphasized that the ROM method does not invoke any linear stability arguments or perturbative expansions. It is clear that the spirit of this method is not limited to systems liable of an exponential fitting procedure. Provided an appropriate fitting function is found and that it is sufficiently simple to be analytically tractable, any system can, in principle, take advantage of the method we have presented here.

Appendix: Study of the function F(x) = x − ln(x)

The problem addressed here, is the determination of approximate solutions to the nonlinear equation:

$$x - \ln(x) - \alpha = 0. \tag{A.1}$$

A.1 General study of F(x)

For x real, $F(x)$ is defined in the interval $]0,\infty[$. Its derivative $f'(x) = 1 - 1/x$ shows that $F(x)$ decays from $+\infty$ to 1 for $0 < x < 1$ and grows from 1 to $+\infty$ for $1 < x < +\infty$. At the minimum, $F(1) = 1$. $F(x)$ is presented in Figure 7. Equation (A.1) thus does not have any real solution for $\alpha < 1$, has one solution for $\alpha = 1$ (*i.e.* $x = 1$) and admits two solutions for $\alpha > 1$.

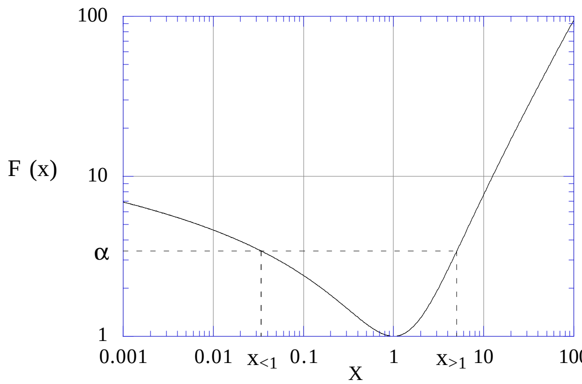

Fig. 7. Representation of $F(x) = x - \ln(x)$ on a log-log scale.

In the case $\alpha > 1$, one of the solutions is smaller than one and the other is greater than one.

A.2 Approximate solution of the nonlinear equation $x - \ln(x) - \alpha = 0$ for $x \gg 1$

In the limit $x \gg 1$, it is worth writing equation (A.1) in the new form:

$$x\left(1 - \frac{\ln(x)}{x}\right) = \alpha. \tag{A.2}$$

Using the fact that $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$, we get a first approximation of the solution, namely $x_0 = \alpha$. This approximation can be improved, using $x_1 = x_0(1+\epsilon)$ where ϵ is considered to be small compared to 1. We can then expand $\ln(x_1)$ in ϵ and get new approximations. At the first order in ϵ, $\ln(x_1) = \ln(x_0) + \epsilon$ and using the first approximation $x_0 = \alpha$, we obtain a new estimation of the root:

$$x_1 = x_0\left(1 + \frac{\ln(x_0)}{x_0 - 1}\right) \quad \text{where} \quad x_0 = \alpha. \tag{A.3}$$

At the next order in ϵ, $\ln(x_2) = \ln(x_0) + \epsilon - \frac{\epsilon^2}{2}$, this method leads to:

$$x_2 = x_0\left(1 + (x_0 - 1)\left[\sqrt{1 + \frac{2\ln(x_0)}{(x_0-1)^2}} - 1\right]\right) \quad \text{where} \quad x_0 = \alpha. \tag{A.4}$$

These approximations satisfy the limits $\lim_{x\to\infty} x_2 = x_1$ and $\lim_{x\to\infty} x_1 = x_0$. We can compare these approximations to the one obtained by a numerical integration of (A.1) using a Newton's method. To make this comparison, we define the error function as:

$$\text{Error}_i(\alpha) = 100\frac{|x_i(\alpha) - x_{Newton}(\alpha)|}{x_{Newton}(\alpha)}. \tag{A.5}$$

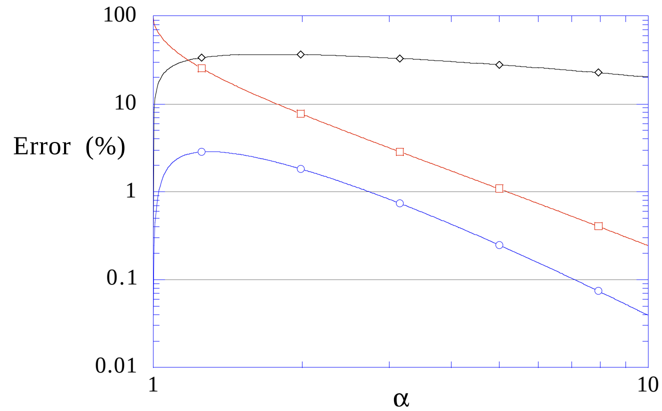

Fig. 8. Error_i as a function of α for $x > 1$ (Error_0 –◇–, Error_1 –□–, Error_2 –○–).

This error function measures in percentile, the discrepancy observed between the numerical solution and the approximations of different orders (i). The functions Error_0, Error_1 and Error_2 are plotted in Figure 8 for $\alpha \in [1, 10]$. We observe that the maximum error obtained with the second order approximation is 2.8%, eventhough the value of α goes down to 1. This accuracy is sufficient for our study.

A.3 Approximate solution of the nonlinear equation $x - \ln(x) - \alpha = 0$ for $x \ll 1$

In this limit, we define a new variable $z = 1/x$ so that $z \gg 1$. In terms of z, equation (A.1) takes the form:

$$\ln(z)\left(1 + \frac{1}{z\ln(z)}\right) = \alpha. \tag{A.6}$$

Using the fact that $\lim_{z\to\infty} \frac{1}{z\ln(z)} = 0$ we obtain the first approximation of the solution $z_0 = e^{\alpha}$. We then pose $z_1 = z_0(1+\epsilon)$ and develop in power of ϵ the two functions $\frac{1}{z_1}$ and $\ln(z_1)$. At the first order in ϵ, we get:

$$z_1 = z_0\left(1 - \frac{1}{z_0 - 1}\right) \quad \text{where} \quad z_0 = e^{\alpha}. \tag{A.7}$$

In the same way, the second order approximation is:

$$z_2 = z_0\left(1 + \frac{z_0 - 1}{2 - z_0}\left[\sqrt{1 - \frac{2(2-z_0)}{(z_0-1)^2}} - 1\right]\right) \quad \text{where} \quad z_0 = e^{\alpha}. \tag{A.8}$$

These approximations satisfy the limits $\lim_{z\to\infty} z_2 = z_1$ and $\lim_{z\to\infty} z_1 = z_0$. In terms of x, we obviously obtain the three approximations $x_0 = 1/z_0$, $x_1 = 1/z_1$ and $x_2 = 1/z_2$. Using the same error functions as the one defined in the

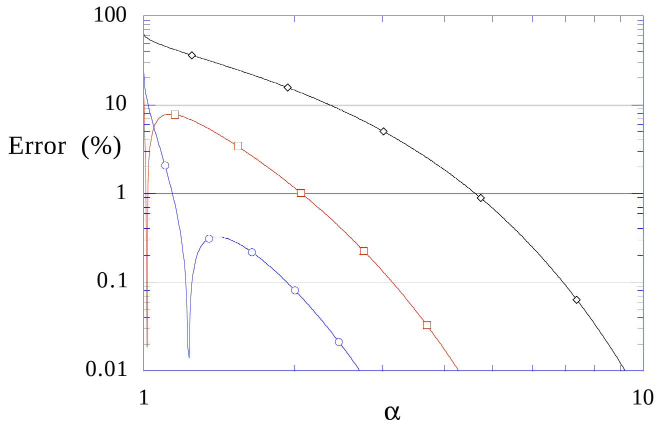

Fig. 9. Error_i as a function of α for $x < 1$ (Error_0 –◇–, Error_1 –□–, Error_2 –○–).

preceding section, we measure the accuracy of these different approximations by comparing them to the numerical solution. This comparison is presented in Figure 9: The maximum error is obtained for $\alpha = 1$, as expected. It reaches 63% for x_0, 12% for x_1 and 22% for x_2. Considering the second order approximation, this error is reduced very quickly to 10% for $\alpha = 1.018$ and to 1% for $\alpha = 1.13$. Considering the first order approximation, the error is always smaller than 12% but decays less rapidly as α increases (it reaches 1% for $\alpha = 2.05$).

To conclude this Appendix, we point out that the approximate solutions obtained for the nonlinear equation $x - \ln(x) = \alpha$ in the two domains $]0, 1[$ and $]1, \infty[$ can be used up to $x = 1$. In other words, we found accurate approximations of this nonlinear equation for the whole domain $x \in]0, \infty[$.

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